§2.4—Product & Quotient Rules

- $f(x)$ is the *y*-value generating "machine."
- $f'(x)$ is the slope value generating "machine."

The *INCORRECT* Product Rule

The derivative of a product of two functions *f* and *g* is the product of the derivatives of *f* and *g*.

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f'(x)\cdot g'(x)$$

The *CORRECT* Product Rule

The derivative of a product of two functions *f* and *g* is .

$$\frac{d}{dx}\left[f(x)\cdot g(x)\right]=f'(x)\cdot g(x)+f(x)\cdot g'(x)$$

Take turns, one derivative at a time per term. The number of factors will equal the number of terms. Multiplication and addition are both commutative, so this one is difficult to mess up, unless you do it incorrectly.

Say it with me, with cadence and conviction: **"*f* prime *g*, plus *f* *g* prime :||"**

Example 1:

Find the derivative of $g(x)=x\left(5x-3x^2\right)$ using the product rule, then verify using the power rule.

Example 2:

Evaluate the following.

(a) $\frac{d}{dx}\left[5x^3\cos x\right]=$

b. $\frac{d}{dx}\left[2x\cos x-2\sin x\right]=$

The *INCORRECT* Quotient Rule

The derivative of a quotient of two functions *f* and *g* is the quotient of the derivatives of *f* and *g*.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)}{g'(x)},\ g(x)\neq 0,\ g'(x)\neq 0$$

The *CORRECT* Quotient Rule

The derivative of a quotient of two functions *f* and *g* is

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2},\ g(x)\neq 0$$

Subtraction is not commutative, so this one is more important to get straight. An easier way to remember it is to think of the numerator as the "**HI**" function (it *is* up high, after all), and the denominator as the "**LO**" function (since it's in the bottom.) To make it more alliterative, let's actually call the denominator the "**HO**" function (it *does* rhyme with "LO.") Finally, let "**d**" mean "the derivative of . . ."

The Quotient Rule now becomes

The HOdHI (Quotient) Rule

The derivative of a quotient of two functions HI and HO is

$$\frac{d}{dx}\left[\frac{HI}{HO}\right]=\frac{HOdHI-HIdHO}{HO\cdot HO},\ HO\neq 0$$

Think of the Seven Dwarfs. Think of a Dyslexic Mr. Wilson greeting Tim. Think of Santa Clause in a hurry.



Example 3:
Find the derivative of each of the following:

(a) $y=\dfrac{4x-2}{x^2-1}$ (b) $f(x)=\dfrac{\sin x}{\cos x}$

Example 4:

Find an equation of the tangent line to the graph of $f(x)=\dfrac{3-(1/x)}{x+5}$ at $x=-1$

Example 5:

Evaluate each of the following.

(a) $\dfrac{d}{dx}\left[\dfrac{x\sin x}{x+1}\right]=$

(b) $\dfrac{d}{dx}\left[\left(\dfrac{x}{x+1}\right)(\sin x)\right]=$

Derivatives of the other Trigonometric Functions

$$\frac{d}{dx}[\tan x]=\sec^2 x \qquad \frac{d}{dx}[\cot x]=-\csc^2 x$$
$$\frac{d}{dx}[\sec x]=\sec x\tan x \qquad \frac{d}{dx}[\csc x]=-\csc x\cot x$$

MEMORIZE THESE IF YOU HAVEN'T ALREADY

Example 6:

Find the derivatives of each of the following using your memory.

(a) $g(x)=x-\tan x$

(b) $y=x\sec x$

Example 7:
Differentiate each of the following functions, then show that they have equivalent derivatives.

(a) $f(x)=\dfrac{1-\cos x}{\sin x}$ (b) $g(x)=\csc x-\cot x$

Higher-Order Derivatives

Recall the following relations among the following three functions.

$s(t)$	position function
$v(t)=s'(t)$	velocity function
$a(t)=v'(t)=s''(t)$	acceleration function

The notation $s''(t)$ is called the **second derivative** of $s(t)$ and we can read it as "***s* double prime of *t***." The second derivative is an example of a higher-order derivative. Why stop at two???

First derivative:	y'	$f'(x)$	$\dfrac{dy}{dx}$	$\dfrac{d}{dx}\left[f(x)\right]$
Second derivative:	y''	$f''(x)$	$\dfrac{d^2y}{dx^2}$	$\dfrac{d^2}{dx^2}\left[f(x)\right]$
Third derivative:	y'''	$f'''(x)$	$\dfrac{d^3y}{dx^3}$	$\dfrac{d^3}{dx^3}\left[f(x)\right]$
Fourth derivative:	$y^{(4)}$	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$	$\dfrac{d^4}{dx^4}\left[f(x)\right]$
$\vdots$				
nth derivative:	$y^{(n)}$	$f^{(n)}(x)$	$\dfrac{d^ny}{dx^n}$	$\dfrac{d^n}{dx^n}\left[f(x)\right]$

Example 8:

(a) If $y = x^3 - 5x^2 + 6x - 4$, find $y^{(5)}(x)$.

(b) $f(x) = 32x^{57} - 25x^{43} + 11x^{27} - 15x^{13} + 9x^3 - 8$, find $f^{(58)}(x)$.

Example 9:

Evaluate:

(a) $\dfrac{d^{447}}{dx^{447}}[\sin x] =$

(b) $\dfrac{d^{8675309}}{dx^{8675309}}[\cos x] =$

Example 10:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-1	3	4	-2	2
0	2	-3	5	-1

The table above gives values for two differentiable functions and their derivatives at selected values of x. Use the table to evaluate the following.

(a) $h'(0)$ if $h(x) = \dfrac{f(x)}{g(x)}$

(b) $h'(-1)$ if $h(x) = x \cdot f(x) \cdot g(x)$

Example 11:

True Story: The position function for a falling object on (or near, ≈ 6 feet high) the surface of the moon is given by $h(t) = -2.66t^2 + h_0$, where $h(t)$ is the height in feet at t seconds and h_0 is the initial height, in feet.

(a) If you were eating at an Italian restaurant on the moon (great food, but no atmosphere), and your meatball rolled off the 3.2-foot high table, after how many seconds would it hit the floor? How fast was the meatball travelling when it hit the ground? If you retrieved it within 5 seconds, would you still eat it?

(b) Using the given equation, determine how many times stronger the acceleration due to gravity on Earth is than that on the moon.

(c) Based on your answer to (b), determine what your weight on the moon would be? Diet problem solved?